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Propagation of discontinuities in a hot relativistic plasma in intense magnetic fields

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Abstract. The growth of discontinuities in a very hot relativistic plasma at temperatures of 10^5 K or above in intense magnetic fields has been studied. The effects of radiation pressure and radiation energy density have been taken into account, while the profiles structured by the radiant heat transfer are assumed to be embedded in the discontinuities. The modes of propagation of weak MHD waves have been determined. The fundamental growth equation governing the growth and decay of weak wavefronts propagating into a hot relativistic plasma in the presence of transverse magnetic fields has been obtained and solved. The relativistic results are found to be in full agreement with the earlier results of classical magnetohydrodynamics in the Newtonian limit. The curvature effects on the global behaviour of wave amplitude have been investigated and a finite time is determined for the formation of caustics due to focusing. The problem of the breakdown of weak waves and the consequent formation of shock waves has also been completely solved and a finite critical time t_c is determined when a weak wave will terminate into a shock wave due to non-linear steepening. The critical amplitude of the initial wave has been determined such that any compressive wave with an initial amplitude greater than the critical one always develops into a shock wave, while an initial amplitude less than the critical one results in a decay of the wave. The relativistic, magnetic and radiation effects on the global behaviour of weak discontinuities have also been investigated and illustrated graphically.

1. Introduction

The recent investigations in astronomy and astrophysics have necessitated the study of relativistic magnetohydrodynamics and, in particular, that of non-linear wave propagation due to its several astrophysical applications (Johnson and McKee 1971, McKee and Colgate 1973, Greenberg 1975). The theory of relativistic fluids and plasmas plays an important role in theoretical astrophysics. Magnetohydrodynamical shock waves appear in the physics of the sun, of the solar system and also of the galaxies. We deem our analysis important for the interpretation of phenomena connected with stellar objects such as collapsed stars and neutron stars, because they possess magnetic fields of very high intensity ($\geqslant 10^{10}$ G) frozen into the matter characterised by very high densities and temperatures.

An extensive body of information on relativistic shocks is available in the published literature. Taub (1948) presented a theoretical foundation of relativistic shock waves. Hoffman and Teller (1950) developed an elegant relativistic treatment of MHD shocks. The pioneering works concerning wave propagation in relativistic MHD flows were done by Lichnerowicz (1967a, b, 1969) and Choquet-Bruhat (1960) in the late fifties and early sixties. Taussing (1973) studied the problem of shock wave production in relativistic plasmas. Massani *et al* (1967) discussed the behaviour of relativistic shock waves in a hot plasma in intense magnetic fields. The weak waves are of particular interest because they are a special class of non-linear wave processes which can be treated rigorously by analytical methods. The explicit results of the analysis give some insight into the interaction of various mechanisms participating in the wave propagation. Kanwal (1966), McCarthy (1969) and Gopalkrishna (1977) studied the problem of growth and decay of relativistic hydrodynamical weak waves in perfect gases. Greco (1975) studied the conditions under which shocks do not form in spite of the quasi-linear hyperbolic systems of the basic equations. The detailed synthesis on the qualitative behaviour of the amplitude of relativistic weak MHD wavefronts was presented by Lichnerowicz (1971) using the distribution theory of generalised functions. Recently, Maugin (1978) studied the propagation of infinitesimal discontinuities in relativistic magnetoelastic media and discussed the conditions for the formation of shock waves. More recently, Ram *et al* (1980) and Ram and Singh (1980) discussed the growth and decay of weak discontinuities in high-temperature phenomena and chemically reacting relativistic fluids. The main academic interest in the present communication is to study the problem of growth and decay of weak waves in relativistic hot plasma in intense magnetic fields by using the singular surface theory of Maugin (1976) and to determine a critical stage when there occurs the breakdown of a weak wave and the consequent formation of a shock wave due to non-linear steepening.

Pai (1966) has suggested that when the mean free path of radiation is very small, the radiative heat transfer term can be neglected except in the boundary layer region but the radiation stresses must be taken into account. In this paper we assume that the mean free path of radiation is small enough to account for the radiation pressure and radiation energy, and the profiles structured by the radiant heat transfer are embedded in the discontinuity; pair formation, thermonuclear reactions and neutrino emissions are negligible (Massani *et al* 1967), as their effects on the shock propagation are negligibly small.

2. Basic preliminaries

The notations used in this paper are, with a few minor exceptions, identical with those employed by Grot and Eringen (1966). Let V_4 be Einstein–Riemann space described by four coordinates

$$x^\alpha = (x^k, x^4) \qquad x^4 = ct$$

and equipped with a normal hyperbolic metric $\mathrm{d}s^2$ (signature +2) expressible in the form

$$\mathrm{d}s^2 = g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$$

where $g_{\alpha\beta}$ is the metric tensor with constant components, t is the time and c is the velocity of light in vacuum. The field of world velocity U^α, expressed as

$$U^\alpha(x^\beta) = \beta(v^k/c, 1) \qquad g_{\alpha\beta}U^\alpha U^\beta = -1$$

where

$$v^k = c\,\partial x^k/\partial x^4 \qquad \beta = (1 - v^2/c^2)^{-1/2},$$

defines the invariant derivative

$$D \equiv U^\alpha \partial_\alpha = (\beta/c)(\partial/\partial t + v^i \partial_i) \qquad \partial_\alpha = \partial/\partial x^\alpha.$$

Here the range of latin indices is 1, 2, 3 and that of greek indices is 1, 2, 3, 4. A dummy index will usually imply summation unless specified otherwise.

The equations governing the motion of a relativistic thermodynamically perfect radiating magnetofluid of infinite electrical conductivity with a constant magnetic permeability μ can be written in the form

$$(\rho U^\alpha)_{,\alpha} = 0 \tag{2.1}$$

$$T^{\alpha\beta}_{,\beta} = 0 \tag{2.2}$$

$$(U^\alpha h^\beta - h^\alpha U^\beta)_{,\beta} = 0 \qquad U^\alpha h_\alpha = 0 \tag{2.3}$$

where

$$T^{\alpha\beta} = \omega U^\alpha U^\beta + (p + p^{\mathrm{R}} + \tfrac{1}{2}\mu h^2) S^{\alpha\beta} - \mu h^\alpha h^\beta$$

$$S^\alpha_\beta = U^\alpha U_\beta + g^\alpha{}_\beta$$

$$\omega = \rho c^2 (1 + e/c^2 + E^{\mathrm{R}}/\rho c^2 + \mu h^2/2\rho c^2)$$

$$h^2 = h^\alpha h_\alpha.$$

Here p, ρ, $T^{\alpha\beta}$, e, h, E^{R} and p^{R} respectively represent the fluid pressure, the matter density per unit of proper volume, the total energy momentum tensor, the internal energy per unit mass of the plasma, the magnetic field intensity, the radiation energy and radiation pressure. A comma followed by an index denotes partial differentiation with respect to the corresponding coordinate. In an optically thick medium under consideration, E^{R}, p^{R} and T are connected by the relation

$$E^{\mathrm{R}} = 3p^{\mathrm{R}} = \alpha_{\mathrm{R}} T^4$$

where T is the absolute temperature and α_{R} is the Stefan–Boltzmann constant.

Projecting (2.2) along and perpendicular to U^α, we obtain

$$c^2 \rho\sigma D U^\alpha + S^{\alpha\beta}(p^* + \tfrac{1}{2}\mu h^2)_{,\beta} - \mu U^\alpha U_\gamma h^\gamma_{,\beta} h^\beta - \mu h^\alpha h^\beta_{,\beta} - \mu h^\alpha_{,\beta} h^\beta = 0 \tag{2.4}$$

$$\{(c^2 \rho\sigma) U^\beta\}_{,\beta} - U^\beta (p^* + \tfrac{1}{2}\mu h^2)_{,\beta} + \mu U_\gamma h^\gamma_{,\beta} h^\beta = 0 \tag{2.5}$$

where

$$p^* = p + p^{\mathrm{R}} \qquad \sigma = f + \mu h^2/\rho c^2$$
$$f = 1 + i^*/c^2 \qquad i^* = e + E^{\mathrm{R}}/\rho + p^*/\rho. \tag{2.6}$$

The equations (2.4) and (2.5) respectively represent the conservation of momentum and energy in relativistic magnetoplasma flows with radiation. Here i^* is the specific enthalpy and f is the index of the radiating plasma. The proper temperature T and the specific entropy η^* of the radiating fluid satisfy the thermodynamic relation

$$c^2\, \mathrm{d}f = T\, \mathrm{d}\eta^* + \mathrm{d}p^*/\rho. \tag{2.7}$$

From (2.3) and (2.4) it follows that

$$U_\alpha U^\beta h^\alpha_{,\beta} + h^\beta_{,\beta} = 0 \tag{2.8}$$

$$h^2 U^\beta_{,\beta} + \tfrac{1}{2} D h^2 - h_\alpha h^\beta U^\alpha_{,\beta} = 0 \tag{2.9}$$

$$c^2\rho f h_\alpha DU^\alpha + h^\alpha p^*_{,\alpha} = 0 \tag{2.10}$$

$$U_\alpha h^\alpha_{,\beta} + h_\alpha U^\alpha_{,\beta} = 0. \tag{2.11}$$

In view of (2.1), (2.7) and (2.9), the equation (2.5) leads to

$$D\eta^* = 0 \tag{2.12}$$

which shows that the motion of the fluid under consideration will be isentropic.

From (2.6), (2.7) and (2.12) we obtain

$$[1+12(\gamma-1)R_p]Dp - (p/\rho)[\gamma+16(\gamma-1)R_p]D\rho = 0 \tag{2.13}$$

where $R_p = p^R/p$ and $\gamma = c_p/c_v$ are the radiation pressure number and heat exponent of the plasma respectively.

3. Compatibility conditions on a time-like singular hypersurface

Let $\Sigma(x^\mu)$ be a time-like regular hypersurface of the space V_4 with parametric equations

$$x^\mu = \psi^\mu(b^\tau) \qquad \tau = 1, 2, 3$$

where b^τ are parametric coordinates of the surface. The surface Σ may be regarded as a surface $S(t)$ in space–time for which the parametric equations are

$$x^i = x^i(b^1, b^2, x^4) \qquad x^4 = ct.$$

If N_α are the components of unit normal vector to $\Sigma(x^\mu)$, n_i the components of the unit space normal to $S(t)$ and G is its speed of propagation, then (Thomas 1963)

$$N^\alpha = \bar{\beta}\{n^i, G/c\} \qquad N_\alpha = \bar{\beta}\{n_i, -G/c\} \tag{3.1}$$

where

$$\bar{\beta} = (1 - G^2/c^2)^{-1/2}.$$

By a weak wave or a weak discontinuity we mean a time-like singular hypersurface $\Sigma(x^\mu)$ across which the flow parameters are continuous, but their first and higher partial derivatives undergo finite jumps at the surface. If $[Z]$ denotes the jump in any flow parameter Z across Σ, then the compatibility conditions due to Maugin (1976) and Truesdell and Toupin (1960) to be satisfied across the weak wave surface $\Sigma(x^\mu)$ can be expressed as

$$[Z_{,\alpha}] = CN_\alpha \qquad [DZ] = VC \tag{3.2}$$

$$\left.\begin{aligned} &[Z_{,\alpha\beta}] = \bar{C}N_\alpha N_\beta + 2N_{(\alpha}x^\tau_{\beta)}C_{;\tau} - CB_{\tau\phi}x^\tau_{(\alpha}x^\phi_{\beta)} \\ &[D(Z_{,\alpha})] = V[Z_{,\alpha\beta}N^\beta] + \delta[Z_{,\alpha}] \end{aligned}\right\} \tag{3.3}$$

where

$$C = [Z_{,\alpha}]N^\alpha \qquad \bar{C} = [Z_{,\alpha\beta}]N^\alpha N^\beta$$

$$b_{\tau\phi} = N_\alpha x^\alpha_{;\tau\phi} \qquad x^\tau_\beta = g_{\alpha\beta}a^{\tau\phi}x^\alpha_{;\phi}$$

$$M_{(\alpha\beta)} = \tfrac{1}{2}(M_{\alpha\beta} + M_{\beta\alpha}) \qquad a_{\tau\phi} = g_{\alpha\beta}x^\alpha_{;\tau}x^\beta_{;\phi}$$

$$V = U^\alpha N_\alpha = -\beta\bar{\beta}G_0/C \qquad G_0 = G - v^i n_i.$$

Here $a_{\tau\phi}$ and $b_{\tau\phi}$ are, respectively, the components of the first and second fundamental covariant tensors of Σ; G_0 is the local speed of propagation of the surface $S(t)$; δ is the generalised form of the δ_t derivative of Thomas and a semicolon denotes covariant derivative with respect to $a_{\tau\phi}$. Moreover, in the local instantaneous rest frame, it can be shown that

$$c\delta[Z_{,\alpha}] = \bar{\beta}^2 \frac{\delta}{\delta t}[Z_{,\alpha}].$$

4. Modes of propagation

Using compatibility conditions for the jumps of inner parts of the equations (2.1), (2.3), (2.4), (2.8)–(2.11) and (2.13), we obtain

$$\rho\lambda^\alpha N_\alpha + V\nu = 0 \tag{4.1}$$

$$V\varepsilon^\alpha + h^\alpha\lambda^\beta N_\beta - \lambda^\alpha h_n - U^\alpha\varepsilon^\beta N_\beta = 0 \tag{4.2}$$

$$c^2\rho\sigma V\lambda^\alpha + \{(1+4R_p)\xi - 4R_p p\nu/\rho + \mu\varepsilon^\mu h_\mu\}S^{\alpha\beta}N_\beta$$
$$- \mu U^\alpha U_\mu \varepsilon^\mu h_n - \mu\varepsilon^\alpha h_n - \mu h^\alpha \varepsilon^\beta N_\beta = 0 \tag{4.3}$$

$$\varepsilon^\alpha U_\alpha V + \varepsilon^\alpha N_\alpha = 0 \tag{4.4}$$

$$h^2\lambda^\alpha N_\alpha + V\varepsilon^\alpha h_\alpha - \lambda^\alpha h_\alpha h_n = 0 \tag{4.5}$$

$$c^2\rho f V\lambda^\alpha h_\alpha + \xi h_n = 0 \tag{4.6}$$

$$\varepsilon^\alpha U_\alpha + \lambda^\alpha h_\alpha = 0 \tag{4.7}$$

$$\xi - (a_R^2 + 4R_p p/\rho)\nu/(1+4R_p) = 0 \tag{4.8}$$

where

$$\lambda^\alpha = [U^\alpha_{,\beta}]N^\beta \qquad \xi = [p_{,\beta}]N^\beta \qquad \nu = [\rho_{,\beta}]N^\beta$$
$$\varepsilon^\alpha = [h^\alpha_{,\beta}]N^\beta \qquad h_n = h^\alpha N_\alpha$$
$$a_R^2 = (p/\rho)\{\gamma + 20(\gamma-1)R_p + 16(\gamma-1)R_p^2\}/\{1+12(\gamma-1)R_p\}.$$

Equations (4.1)–(4.8) can be combined in the form

$$(A^\alpha_\beta - \phi\delta^\alpha_\beta)\lambda^\beta = 0 \tag{4.9}$$

where

$$\phi = \sigma c^2 V^2 - \mu h_n^2/\rho$$
$$A^\alpha_\beta = (a_R^2 + \mu h^2/\rho - a_R^2\mu h_n^2/\rho c^2 f V^2)S^{\alpha\gamma}N_\gamma N_\beta + (\mu h_n/\rho)[(a_R^2/fc^2) - 1]h^\alpha N_\beta.$$

The system (4.9) will admit non-trivial solutions iff

$$|A^\alpha_\beta - \phi\delta^\alpha_\beta| = 0 \tag{4.10}$$

which provides us with the following three modes of propagation:

(i) $$G_{0A}^2 = \frac{b_n^2}{\sigma + b_n^2/c^2}$$

(ii) $$G_{0SM}^2 = \frac{a_e^2 - [a_e^4 - 4(\sigma - a_e^2/c^2)a_R^2 b_n^2/f]^{1/2}}{2\beta^2\bar{\beta}^2(\sigma - a_e^2/c^2)}$$

(iii) $$G_{0FM}^2 = \frac{a_e^2 + [a_e^4 - 4(\sigma - a_e^2/c^2)a_R^2 b_n^2/f]^{1/2}}{2\beta^2\bar{\beta}^2(\sigma - a_e^2/c^2)}$$

where

$$b_n^2 = \mu h_n^2/\rho \qquad a_e^2 = a_R^2 + \mu h^2/\rho$$

and G_{0A}, G_{0SM}, G_{0FM} are, respectively, the velocities of Alfvén waves, slow magnetohydrodynamic waves and fast magnetohydrodynamic waves.

These results coincide with those of classical magnetohydrodynamics (Ram and Singh 1977) in the non-relativistic limit ($\sigma = 1$, $b_n^2/c^2 = 0$, $a_R^2/c^2 = 0$, $\beta = \bar{\beta} = 1$). When the magnetic field acts transversely to the direction of propagation, $b_n = 0$ and hence there is only one mode of propagation given by

$$G_0^2 = a_e^2/\beta^2\bar{\beta}^2(\sigma - a_e^2/c^2) \tag{4.11}$$

which on account of (3.5) takes the form

$$V^2 = a_e^2/(c^2\sigma - a_e^2). \tag{4.12}$$

In an instantaneous rest frame, the equation (4.11) assumes the form

$$G_0^2 = a_e^2/\sigma$$

which is in full agreement with earlier results of Ram and Singh (1977) and McCarthy (1969) in particular cases.

5. The growth equation

This section is devoted to the derivation of a fundamental growth equation which will govern the growth and decay of a weak discontinuity during its course of propagation through a hot plasma in a transverse magnetic field. The medium ahead of the wavefront is assumed uniform and at rest with the magnetic field in the frozen state.

Now we define the amplitude b of the wave $\Sigma(x^\mu)$ by the relation

$$b = c\lambda = c\lambda^\alpha N_\alpha = c\lambda^\alpha N_\alpha^* \tag{5.1}$$

where $N^{*\alpha} = S^{\alpha\beta}N_\beta$ are the space-like components of N^α.

Differentiating the equations (2.1), (2.4), (2.8) and (2.9) with respect to x^β and taking jumps across Σ with the help of (3.2) and (3.3), we obtain

$$\begin{aligned}\rho(\sigma + a_e^2/c^2)\delta(\lambda) &+ \rho V[\sigma - a_e^2(1+V^2)/V^2c^2]\bar{\lambda}^\alpha N_\alpha - \rho a_e^2 x_\alpha^\tau \lambda_{;\tau}^\alpha (1+V^2)/Vc^2 \\ &- \rho\sigma\lambda^\alpha\delta(N_\alpha) + (a_e^2/Vc^2)\lambda S^{\alpha\gamma}N_\alpha\delta(\rho N_\gamma/V) + (\rho\mu/Vc^2)\lambda h_\alpha\delta(h^\alpha/\rho) \\ &+ \rho\lambda^2[(\rho g^2 + 2a_e^2 + \mu h^2/\rho)(1+V^2)/V^2c^2 - 3a_e^2/c^2] \\ &= 0 \end{aligned} \tag{5.2}$$

where

$$g^2 = (\partial^2 p^*/\partial\rho^2)_{\eta^* = \text{constant}}.$$

In view of (4.12), the coefficient of $\bar{\lambda}^{\alpha}N_{\alpha}$ in (5.2) vanishes and, therefore, we obtain the following equation to be satisfied by λ

$$
\begin{aligned}
\rho(2\sigma - a_e^2/c^2)\delta(\lambda) - (\rho a_e^2/Vc^2)[x_{\alpha}^{\tau}N^{*\alpha}\lambda_{;\tau} + \lambda N^{*\alpha}_{,\alpha}] \\
+ (a_e^2/Vc^2)\lambda S^{\alpha\gamma}N_{\alpha}\delta(\rho N_{\gamma}/V) - \rho\sigma(1+V^2)^{-1}\lambda N^{*\alpha}\delta(N_{\alpha}) \\
+ \rho\lambda^2[(\rho g^2 + \mu h^2/\rho + 2a_e^2)(1+V^2)/V^2c^2 - 3a_e^2/c^2] \\
+ (\rho\mu\lambda/Vc^2)(1+V^2)h_{\alpha}\delta(h^{\alpha}/\rho) \\
= 0 \qquad (5.3)
\end{aligned}
$$

which is the required growth equation governing the global behaviour of the amplitude $c\lambda$ of a relativistic weak wave in a dense plasma in a transverse magnetic field. In a local instantaneous rest frame for which

$$N^{*\alpha} = (1+V^2)^{1/2}(n^i, 0)$$

equation (5.3) takes on a particularly simpler form

$$A\delta b/\delta t - \Omega b + Bb^2 = 0 \qquad (5.4)$$

where

$$
\begin{aligned}
\Omega &= -\partial n^i/2\partial x^i \\
A &= (G_0/2a_e^2)(2\sigma - a_e^2/c^2)/(1 - G_0^2/c^2) \\
B &= (G_0/2a_e^2)[(\rho g^2 + 2a_e^2 + \mu h^2/\rho)\sigma/a_e^2 - 3a_e^2/c^2].
\end{aligned}
$$

Here Ω is the mean curvature of the propagating surface $S(t)$ in space–time.

If s denotes the distance traversed by the wave along its normal trajectory in time t, we have

$$\delta s/\delta t = G_0 \qquad (5.5)$$

which provides us a relation $s = G_0 t$, where G_0 is the constant speed of the wavefront propagating in a uniform state ahead of it in the rest frame of this uniform state. For non-planar waves, $\Omega \neq 0$ and is calculated in Thomas (1963) in the form

$$\Omega = \frac{\Omega_0 - K_0 s}{1 - 2\Omega_0 s + K_0 s^2} \qquad (5.6)$$

where Ω_0 and K_0 are the values of the mean and Gaussian curvatures of the initial wavefront.

Using (5.5) and (5.6) in (5.4) and solving for b in the space–time, we obtain

$$b = b_0 F(t)\left(1 + b_0(B/A)\int_0^t F(t)\,\mathrm{d}t\right)^{-1} \qquad (5.7)$$

where b_0 is the initial wave amplitude at time $t = 0$ and

$$F(t) = [(1 - K_1 G_0 t)(1 - K_2 G_0 t)]^{-1/2AG_0}.$$

Here K_1 and K_2 are principal curvatures of the initial wavefront.

6. Global behaviour of the wave amplitude

We shall first study the effects of wave geometry on the global behaviour of the amplitude $b(t)$. For a converging wave, K_1 and K_2 are positive so that there exists a finite time t^* which is the least positive root of

$$(1-K_1G_0t)(1-K_2G_0t)=0$$

such that when $|b_0|\leqslant b_c$,

$$\lim_{t\to t^*} b(t)=\infty$$

where

$$b_c=\left((B/A)\int_0^{t^*} F(t)\,\mathrm{d}t\right)^{-1}. \tag{6.1}$$

In this case a converging wave will form a cusp or caustic due to focusing after a finite time t^* depending on the curvatures K_1 and K_2.

If $|b_0|>b_c$, there will exist a finite critical time $t_c<t^*$ given by

$$\int_0^{t_c} F(t)\,\mathrm{d}t=AG_0/B|b_0| \tag{6.2}$$

such that

$$\lim_{t\to t_c} b(t)=\infty.$$

This shows that the weak wave will turn into a diffracted shock wave due to non-linear steepening after a finite critical time $t_c<t^*$.

In the case of a diverging wave, K_1 and K_2 are negative and hence no cusp will be formed. But there exists a critical value b_c of the initial wave amplitude $b_0(b_0<0)$ for a compressive wave such that

(i) when $|b_0|<b_c$, $\lim_{t\to\infty} b(t)=0$;
(ii) when $|b_0|>b_c$, $\lim_{t\to t_c} b(t)=\infty$,

where

$$b_c=\left((B/A)\int_0^{\infty} F(t)\,\mathrm{d}t\right)^{-1}. \tag{6.3}$$

Thus we conclude that compressive waves with $|b_0|>b_c$ will grow into a shock wave formed after a finite time t_c given by (6.2) and those with $|b_0|<b_c$ will ultimately decay.

From (6.2) and (6.3), we have

$$\frac{\mathrm{d}t_c}{\mathrm{d}|K_1|}=\left(2AF(t_c)\right)^{-1}\int_0^{t_c} F(t)(1+|K_1|G_0t)^{-1}\,\mathrm{d}t>0 \tag{6.4}$$

$$\frac{\mathrm{d}b_c}{\mathrm{d}|K_1|}=(Bb_0^2/2A)\int_0^{\infty} F(t)(1+|K_1|G_0t)^{-1}\,\mathrm{d}t>0. \tag{6.5}$$

Equations (6.4) and (6.5) show that the critical time t_c and the critical amplitude b_c increase with curvature effects.

7. Special cases

Case I. In order to investigate relativistic and magnetic field effects on the global behaviour of the wave amplitude $b(t)$, we shall first study the case of a plane wave for which $\Omega = 0$. In the non-radiative case ($R_p = 0$) the solution (5.7) takes a simple form

$$b = b_0(1 - BT/A)^{-1} \tag{7.1}$$

where

$$A = \frac{\{1+\tau[N_A + (\gamma-1)^{-1}]\}^{1/2}\{2+\tau[N_A + (3-\gamma)/(\gamma-1)]\}}{2a(1+N_A)^{1/2}[1+\tau(2-\gamma)/(\gamma-1)]}$$

$$B = \frac{1+\gamma+3N_A+[\tau(2-\gamma)/(\gamma-1)][(2+N_A)(4-\gamma)]}{2a(1+N_A)^{3/2}\{1+\tau[N_A+(\gamma-1)^{-1}]\}^{1/2}}$$

$$T = \hat{b}_0 t \qquad N_A = \mu h^2/\rho a^2 \qquad \tau = a^2/c^2.$$

Here τ and N_A are dimensionless parameters for the relativistic effect and the magnetic effect, respectively; T is the dimensionless parameter of time. For any practical problem we have $0 \leq \tau < 1$ and $0 \leq N_A < 1$. The relativistic and magnetic field effects on the growth and decay of discontinuities are shown in figures 1–4. Figure 3 shows that the shock formation time increases with relativistic effects. Figures 1, 2, 4 show that in an ultra-relativistic case the magnetic field also delays the shock formation, whereas in the non-relativistic case as well as under low relativistic effects it accelerates the shock formation. This implies that there is a very interesting competition between the

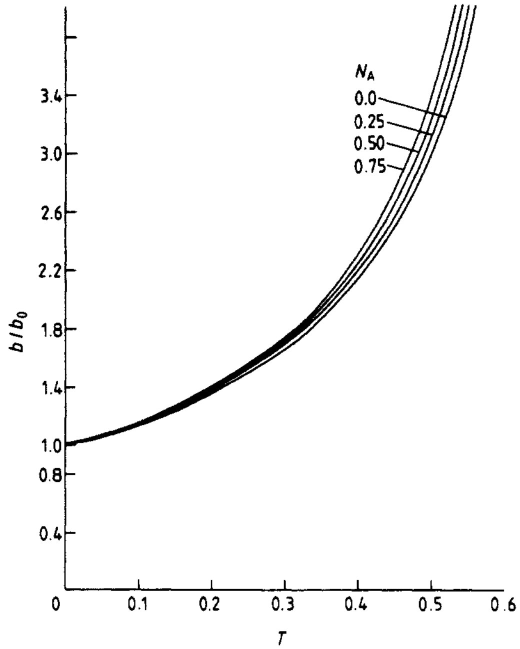

Figure 1. Magnetic field effects on the growth of compressive weak discontinuities in non-relativistic fluids for $\gamma = 5/3$.

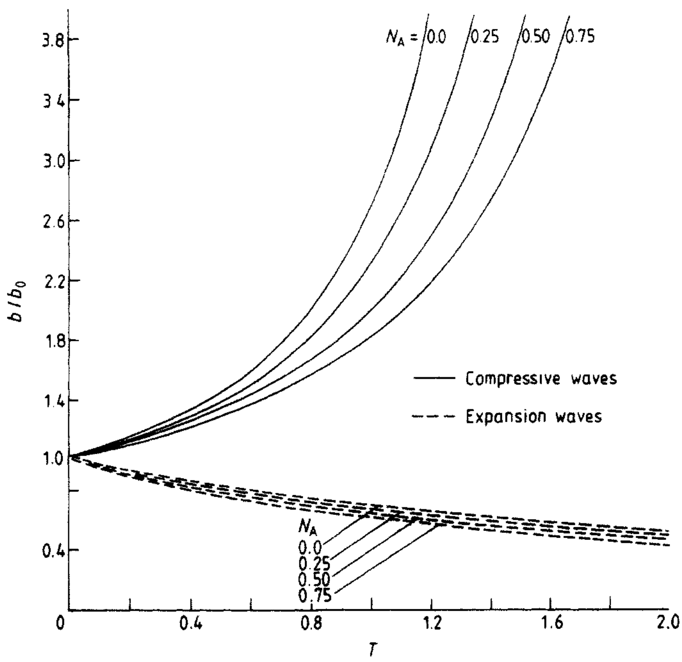

Figure 2. Magnetic field effects on the growth and decay of weak non-linear MHD waves in relativistic fluids for $\gamma = 5/3$ and $\tau = 0.75$.

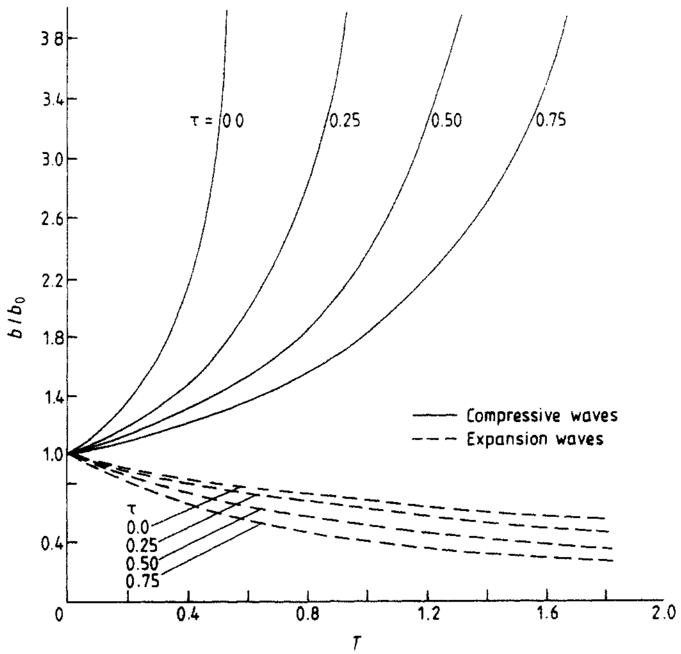

Figure 3. Relativistic effects on the growth and decay of weak non-linear MHD waves for $\gamma = 5/3$ and $N_A = 0.75$.

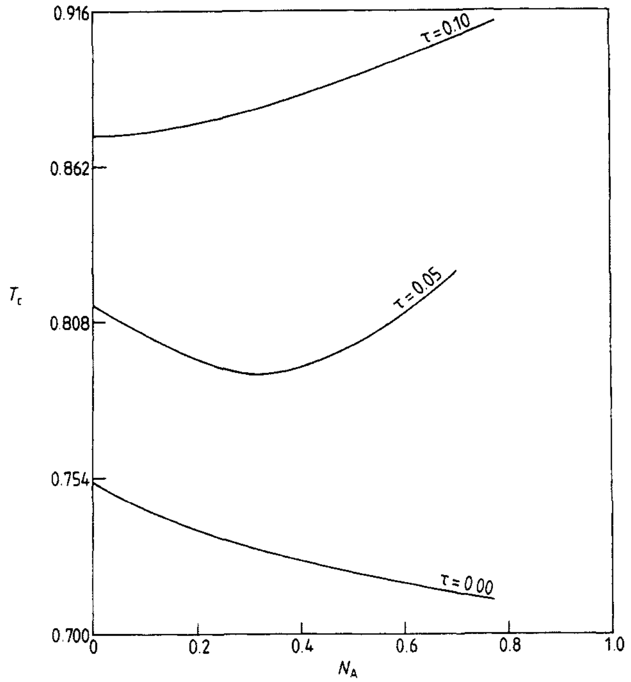

Figure 4. Variation of the shock formation time with respect to the magnetic number under different relativistic effects for $\gamma = 5/3$.

magnetic field effect and relativistic effects. Figure 4 shows an interesting interaction between the relativistic effects and the magnetic field effects on the critical time t_c for the shock formation.

Case II. In order to study relativistic as well as radiation effects on the growth of a compressive wave, we shall now consider the case of a non-planar wave for which $\Omega \neq 0$ and $K_1 = K_2$. In this case the solution (5.7) assumes the form

$$\frac{b}{b_0} = (1-T)^{-1/AG_0}\left[1 + \frac{K}{AG_0 - 1}\left(1 - (1-T)\frac{AG_0 - 1}{AG_0}\right)\right]^{-1} \tag{7.2}$$

where

$$AG_0 = \frac{X - \frac{1}{2}Y}{X - Y} \qquad X = \left[\gamma + \tau\left(\frac{\gamma}{\gamma - 1} + 4R_p\right)\right][(1 + 12(\gamma - 1)R_p]$$

$$Y = \tau[\gamma + 20(\gamma - 1)R_p + 16(\gamma - 1)R_p^2]$$

$$K = b_0 B/K_1 \qquad T = K_1 G_0 t.$$

Here R_p and K are dimensionless parameters for the radiation effect and the curvature effect respectively; T is the dimensionless parameter of time. For any practical problem we have $0 \leqslant R_p < 1$. The radiation pressure effect on the growth of a weak discontinuity is shown in figure 5. It is evident that the radiation stresses cause a decrease in the critical time t_c for the shock formation and hence accelerate the shock formation process of a compressive wave.

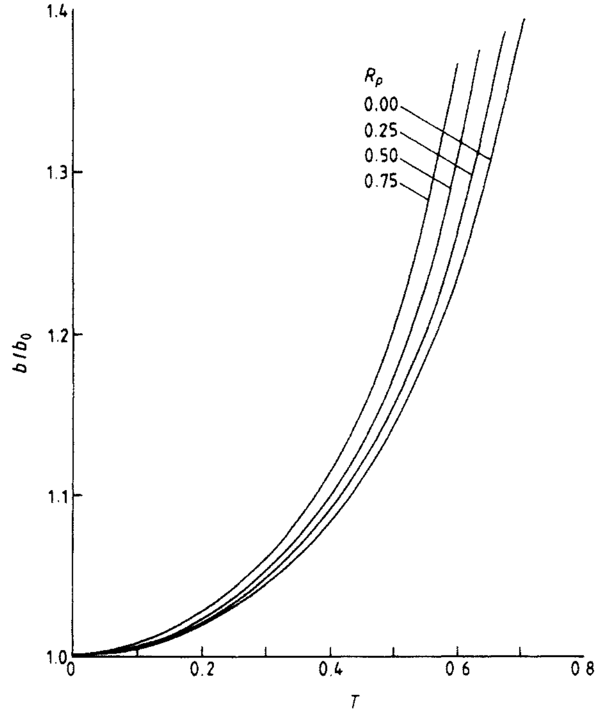

Figure 5. Radiation stress effects on the growth of weak spherical non-linear waves in relativistic fluids for $\gamma = 5/3$ and $\tau = 0.75$.

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